Tentamen Partiële Differentiaalvergelijkingen

30 juni 2008, 9.00-12.00 uur.

Problem 1,2,5: max. 2 points; problem 3,4,6: max. 1 point. Total: 9+1 (free) =10 points. Success!

1. We consider the following PDE

$$xu_x + yu_y = u + 1$$

   (a) Find the solution $u(x, y)$ of this PDE that satisfies the condition $u(x, x^2) = x^2$.

   (b) Give a function $f(x)$ such that the PDE with the condition $u(x, f(x)) = x^2$ cannot be solved by means of the method of characteristics. Explain why the method of characteristics fails.

2. (a) For $T > 0$ let $Q_T = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$. Suppose that $u(x,t)$ satisfies the differential inequality

$$u_t - a(x,t)u_{xx} - b(x,t)u_x < 0$$

   for $(x,t) \in Q_T$ where $a(x,t) \geq 0$ in Q_T. Show that $u(x,t)$ cannot achieve a local maximum in Q_T.

   (b) Show that there is no maximum principle for the wave equation $u_{tt} + c^2 u_{xx} = 0$.

3. Let

$$u_n(x,t) = \frac{1}{n} \sin(nx) e^{-n^2 kt}$$

   (a) Check that u_n satisfies the diffusion equation $u_t = ku_{xx}$ for all x, t.

   (b) Show that $u_n(x, 0) \to 0$ as $n \to \infty$. Now, consider any $t < 0$ and show that $|u_n(x,t)| \to \infty$ as $n \to \infty$ except for a few x ($x = k\pi/n$ with k integer).

   (c) Explain, with the help of (a) and (b), that the diffusion equation is not well-posed for $t < 0$ ("backward in time").

4. Determine the Green's function of the problem

$$\frac{du}{dx} = 0 \qquad x \in (\alpha, \beta)$$

$$u(\alpha) = u_\alpha \qquad u(\beta) = u_\beta$$

Z.O.Z

5. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is approximately modeled by the fourth-order PDE

$$u_{tt} + c^2 u_{xxxx} = 0$$

It has initial conditions as for the wave equation: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ where ϕ and ψ are given functions of x. Let's say that the end $x = 0$ of the bar is clamped (fixed), meaning that $u(0,t) = u_x(0,t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l,t) = u_{xxx}(l,t) = 0$. Thus there are in total four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.

(b) Show that zero it not an eigenvalue.

(c) Show that

$$\lambda = \frac{||X''||^2}{||X||^2}$$

where $||X||^2 = (X,X)$ and $(X,Y) = \int_0^l X(x)Y(x)dx$.

(d) Write the eigenvalues as $\lambda = \beta^4$ and find the equation for β.
Note: The frequencies are $c\beta_n^2$ with $n = 1, 2, \ldots$ Relatively to the fundamental ($n = 1$) frequency the first overtone of the bar is higher than the fifth overtone of a vibrating string. This explains why you hear an almost pure tone when you listen to a tuning fork.

6. Suppose that the maximum principle holds for the parabolic initial, boundary value problem

$$u_t - u_{xx} = F(x,t) \quad \text{for} \quad 0 < x < l, \; 0 < t \le T$$

with $u(0,t) = g(t)$, $u(l,t) = h(t)$ $(0 \le t \le T)$ and $u(x,0) = f(x)$ $(0 \le x \le l)$.
Here all functions are continuous on their domains of definition.
Prove that there exists at most one (continuous) function $u(x,t)$ (defined on $0 \le x \le l, 0 \le t \le T$) that satisfies this parabolic initial, boundary value problem.